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Opto-Electronic Belts for Recording Respiration in Psychophysiological Experimentation and Therapy

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ABSTRACT

A temperature-compensated and a simpler, non-compensated respiration measuring belt based on an infrared light sensing principle are described. The systems are designed to overcome shortcomings in usual respiration monitoring methods applicable to: 1) a respiratory-sinus-arrhythmia quantification study, and 2) a portable circulation monitoring system. Both devices have a rugged and lightweight construction, are comfortable for subjects, relatively free from artifacts, are of low cost, and are simply interfaced to recording apparatus. Additionally, the temperature-compensated version allows for DC-recording, permitting monitoring of breath-holding, which is not feasible with conventional transducers. The simple construction of the non-compensated version makes it especially suitable to routine therapy and monitoring applications. Use of two temperature-compensated belts measuring thoracic and abdominal circumference allows for breath-volume calibration by multiple linear regression with reliabilities between 75 and 95%. The mathematical basis for this calibration procedure is discussed in detail.

DESCRIPTORS: Respiration measurement, Breath-volume, Breath-holding, Biofeedback, Transducers, Light emitting diode.

Respiration measuring devices used in psychophysiology are of two kinds. Those derived from physiological applications give exact recording of respiratory airflow but require bulky apparatus and use of a face mask. Lighter weight devices that are less disturbing for the subject are more adapted to the requirements of psychophysiological and occupational-physiological experimentation and monitoring, but usually give precise registration of only respiration frequency. Most of these latter devices rely on some measurement of thoracic or abdominal circumference, but other principles are also employed.

In the course of developing procedures for estimation of system theory parameters of respiratory sinus arrhythmia, we have tested several methods of circumference recording. Subjects were led to perform paced respiration to produce specific respiratory patterns (so called test functions). One of the main respiratory test functions was a step function: Quick inhalation followed by holding the breath for 20 sec. To accurately measure the respiration the device has to show high repetition reliability, absence of time lags (stemming from differential transfer properties of AC-coupling), and, most importantly, sufficiently low long term drift to allow for DC-coupling. Usually AC-coupling is used to eliminate the problem of long term drifting. But AC-coupling time-constants provided by most physiological amplifier inputs are at most 20 sec, which is too short to accurately measure the signal in the 20-sec breath-holding situation. Finally, because the method was to be routinely used in behavior therapeutic sessions it had to be an easy to use, lightweight, and rugged device, comfortable for the subjects to wear, with low susceptibility to touch artifacts, low extension force, and low price.

Among the tested devices were Beckman's type 7001 respiration transducer[1], and other belts using strain gauges, Stoelting's airfilled rubber tube with connected pressure-transducer[2], a custom built potentiometric device utilizing "conductive plastic" linear resistive tracks[3], standard mercury capillary

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[1]Beckman Instruments, Inc., Schiller Park, Illinois.

[2]C. H. Stoelting Co., Chicago, Illinois 60024.

[3]"Conductive plastic"-tracks are supplied by Penny and Giles, South Wales, England. The device measures 20×24×207 mm, weight is 130 g.

length gauges (Shapiro & Cohen, 1965), capillary length tubes filled with several electrolytic solutions and appropriately adapted input-couplers[4], and a thermistor airflow measurement method[1]. None of these met all our requirements.

The mechanical and electrical shortcomings of the available methods led to the construction of a new respiration measuring belt using infrared light detection for distance calculation. This report describes its construction and properties as well as its applicability as a breath-volume measurement method and use as a feedback device for paced respiration.

Construction of Opto-Electronic Respiration Belts

For construction of the belt a small light emitting diode (LED) and a phototransistor are glued[5] about 10 mm apart onto a 30 mm wide elastic band. Detected light intensity is used as a measure of distance. Although this method results in a quadratic relationship between signal and distance, for the occurring small distance changes we will show that it can be linearly approximated. The whole transducer part is covered with black silicon rubber to protect against changes in ambient light, and the light path underneath is filled with transparent silicon rubber[6] (Figure 1). Opacity of the clear silicon is not critical, but attention should be paid to the choice of a sufficiently elastic material, so that movement is not hindered.

The rubber belt chosen needs an extension force of .18 N/cm (in a range from 7 to 25 cm extension, correlation coefficient between force and extension $r = .999$), and for secure fit we fastened it with approximately 2.5 N. The weight of the whole belt is 60 g. Initially a velcro band was used for fastening, but since it tends to get blocked with lint we changed to ordinary buckles.

Due to the high temperature coefficient of both the LED and the transistor, temperature compensation has to be added if DC-coupling to the recording input is required. Several common compensation techniques use series and shunt resistors across the opto-electronic elements (Irrgang & Stemmler, 1979; Schmidt & Feustel, 1975). We chose a different approach using a second light-emitting/receiving pair with constant distance, mounted near the actual measuring pair. The second light path is wired to the negative feedback loop of the pick-up amplifier. Since it has the same temperature as the measuring pair, the common temperature variations cancel out. This method is superior to others since it accomplishes compensation of both LED and phototransistor simultaneously, so that time-consuming matching of compensation-resistor values to each individual opto-element's temperature characteristic is avoided, and, additionally, less supply current is drawn. Figures 2 and 3 show the circuit diagram of transducer and coupling electronics. Trim pots are initially adjusted to an LED-operating point of several milliamps and zero-volt signal output. The high-impedance highpass filter can be used for long time-constant AC-coupling (see Figure 3).

[4]Modified Beckman coupler for Dynograph R. Circuit diagram is available.

[5]As an adhesive, hot-tempered epoxy has proven successful.

[6]These silicones are supplied by Wacker Chemie, Muenchen, Germany.

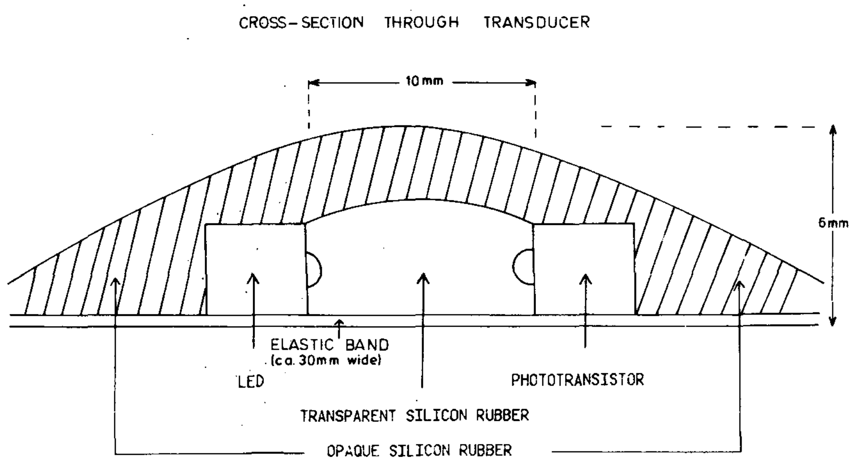

Figure 1. Opto-electronic respiration belt. An infrared-sensitive phototransistor measures light emitted from an infrared light emitting diode. They are mounted on an elastic band used as a belt around chest or abdomen, so their distance varies with breathing. Detected light flow is taken as a measure of circumference.

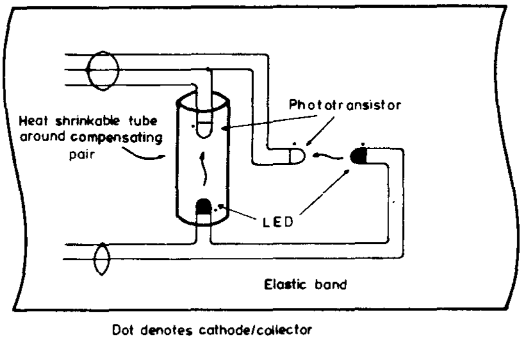

Figure 2. Temperature-compensated opto-electronic respiration belt. To the right, the measurement phototransistor and LED are cemented on the rubber band. To the left, a compensating pair is rigidly fixed in a heat shrinkable tube.

We measured the temperature dependency of the compensated device in a range from 10 to 40°C and found that it is best (least squares fit) described by the following function:

$$U_o = 1003 - 6.355\ T + .0566\ T^2$$

where U_o: Signal output voltage in millivolts, and T : Temperature in degrees Celsius, corresponding to a temperature coefficient of -3.6 mV/°C. Signal voltages of interest are on the order of 50 mV; thus under ordinary conditions temperature dependency can be ignored[7].

Signal output voltage as a function of extension in a range from 8 to 22 cm is described by the relationship

$$U_o = 6.01 - 9.3/s - 310/(s^2)$$

where U_o: Signal output voltage in volts, and s: Extension in centimeters[8]. For measurement the belt was prestretched with a force of 2.5 N; extensions are stated relative to this starting position. Ordinary breathing results in extensions on the order of several millimeters. As can be seen from the above equation, for this range a linear relationship is a sufficient approximation. For extensions from 15 to 16 cm for example, the signal voltage characteristic is given by

$$U_o = .89 + .208 \cdot s\,.$$

Coincidence of data points for equal extensions is a measure of reliability of the measurement. Variation amounts to 4.2% of the output range for 1 cm extension (correlation coefficient $r = .9787$); for a range of 14 cm extension the variation is .13% (correlation with hyperbola, $R = .9993$).

In use the belt can be attached around the body anywhere between abdomen and chest. Which position results in the highest correlation with breath-

[7]If AC-coupling is sufficient a still simpler non-compensated version of the belt is available. Although not as accurate, it is easier to construct and does not need the special coupling electronics of Figure 3, but needs only a supply current for the LED. Signal output voltage of this device is in the range of 10 mV. Its dependency on extension is similar to that of the compensated belt.

[8]The constant term "6.01" represents the DC-offset voltage, which in this case was adjusted so that 8 cm extension corresponded to 0 V signal.

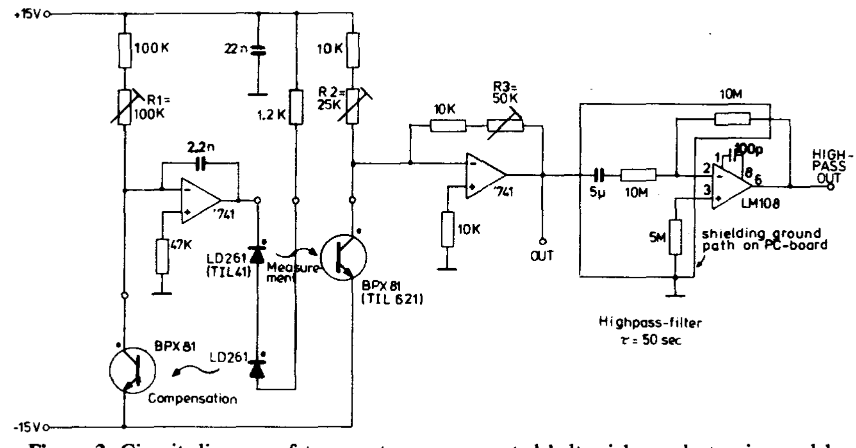

Figure 3. Circuit diagram of temperature-compensated belt, pick-up electronics, and long time-constant highpass filter. The compensating light emitting/receiving pair has a fixed distance and is mounted near the measuring pair. The shown highpass filter employs a high-impedance op-amp to obtain extremely high time constants. For electrostatic shielding high-impedance filter elements are surrounded by a ground path on the circuit-board. Use metal-film resistors for improved temperature stability.

Adjustments: Trim-potentiometer R_1 is used to initially set the LED current to some milliamperes; next, the output voltage is set to 0 volt with R_2; R_3 allows for limited amplitude scaling.

volume is different among subjects, but most people show better predictions of breath-volume with a belt location closely under the arms.

For 15 subjects correlations between expired volume as measured with a mechanical spirometer and belt output signal ranged between .72 and .97 (for optimal belt placement and a breath-volume range from .5 to 2 liters) (Table 1). For calibration, subjects have to be guided to breathe at several different volumes to obtain sufficient variation in the independent variable (recommended range is about .5 to 2 liters but depends on the subject's tidal volume). Again, the relationships were linear in the measured range of breath-volumes.

This simple and low-cost device is comfortable to wear, is insensitive to touch artifacts, and yields highly reliable results. It has been used in a psychophysiological monitoring system for behavior therapeutic stress-management (Rombouts, Muehlberger, Klenk, Bolsinger & Ferstl, Note 1), and as a portable occupational-physiological registration system.[9] Furthermore, combined with the measurement method described below, it has been successfully used in a study investigating the dependency of heart rate variations on respiration. A full description of this study will appear in a separate paper.

Prediction of Breath-Volume from Thoracic and Abdominal Circumferences

Measures of thoracic and abdominal circumferences are not reliable predictors of breath-volume on their own. Although the data presented in Table 1 show good correlations, they are given for optimal belt placement, which has to be determined beforehand. Moreover, when subjects change between "breast breathing" and "abdominal breathing," optimal belt position is lost.

[9]This system is now manufactured by Langer Elektronik, Eching bei Muenchen, Germany.

Table 1

Correlations of expired volumes as measured with a mechanical spirometer and belt signal amplitudes

Subject No.	Maximum Correlation (r)	Relative Error Variance $(1-r^2)$ %	No. of Exhalations (n)	Standard Deviation of Volume (ml)	Belt Position
1	.9746	5.0	50	549	Abdomen
2	.9435	11.0	13	128	Chest
3	.8893	20.9	17	304	Abdomen
4	.7176	48.5	26	139	Chest
5	.9317	13.2	21	345	Abdomen
6	.8011	35.8	22	333	Abdomen
7	.9122	16.8	26	358	Abdomen
8	.9357	12.5	29	388	Abdomen
9	.7750	39.9	39	318	Chest
10	.8249	32.0	31	376	Chest
11	.8386	29.7	36	362	Chest
12	.8008	35.9	27	474	Chest
13	.8632	25.5	15	715	Abdomen
14	.9432	11.0	23	234	Chest
15	.9505	9.7	22	249	Chest

Combinations of both thoracic and abdominal signals can be expected to give more precise results, as, for example, Shapiro and Cohen (1965) describe. Starting from a cylindrical model of the chest they derived a square dependency of volume and circumferences:

$$\Delta V = k_1 \cdot \Delta C_1^2 + k_2 \cdot \Delta C_2^2 \qquad (1)$$

where ΔV: Change in breath-volume, ΔC_1: Change in thoracic circumference, and ΔC_2: Change in abdominal circumference.

To determine the relationship of weights, $n^2 = k_2/k_1$, the subjects were instructed to perform breath-resembling chest movements with nose and mouth shut. From Equation 1 with $\Delta V = 0$ this relationship was:

$$k_2/k_1 = -\Delta C_1^2/\Delta C_2^2 . \qquad (2)$$

The common scaling factor k_1 was determined by regression of predicted volume (Equation 1) to actual volume (as measured with a spirometer) on an analog computer.

In the derivation below, however, we show that one can adequately represent the relationship between change in volume and change in both circumferences by a simpler equation using only first-order terms (and *not* second-order terms as in Equation 1). Assuming a truncated-cone model of the air-cavity (one can also assume a cylindrical model), let C_1 and C_2 be its top and bottom circumferences and let its height be h. Its volume is then given by

$$V = \frac{h}{12\pi}(C_1^2 + C_2^2 + C_1C_2) . \qquad (3)$$

Let ΔC_1 and ΔC_2 be the circumference changes associated with breathing, and let the volume change be ΔV. Then

$$V + \Delta V = \frac{h}{12\pi}((C_1+\Delta C_1)^2 + (C_2+\Delta C_2)^2 + (C_1+\Delta C_1)(C_2+\Delta C_2))$$

which leads to a volume increment

$$\Delta V = \frac{h}{12\pi}((2C_1+C_2)\cdot\Delta C_1 + (2C_2+C_1)\cdot\Delta C_2 + \Delta C_1^2+\Delta C_2^2+\Delta C_1\,\Delta C_2) \qquad (4)$$

or, abbreviated,

$$\Delta V = k_3 \cdot \Delta C_1 + k_4 \cdot \Delta C_2 + \frac{h}{12\,\pi}(\Delta C_1^2 + \Delta C_2^2 + \Delta C_1 \Delta C_2) \qquad (5)$$

with constants $k_3 = (h/12\pi)\ (2C_1 + C_2)$ and $k_4 = (h/12\pi)\ (2C_2 + C_1)$.

Ignoring second-order terms leads to the linear Equation

$$\Delta V = k_3 \cdot \Delta C_1 + k_4 \cdot \Delta C_2 . \qquad (6)$$

The error introduced by ignoring the higher order terms can be shown to be smaller than ½ $(\Delta C_1/C)$, C being the smaller of the two circumferences (for detailed proof see Appendix A). Now respiratory circumference changes ΔC_i amount to about 5% of total circumferences C_i, and, even for a deep breath, never exceed 10%. Ignoring second-order terms thus introduces an error of less than 5%, so that the linear Equation 6 is a sufficient approximation of Equation 5. Air-cavity volume changes—i.e. inhaled or expired breath-volumes—can therefore be calculated simply as the weighted sum of circumference changes.

A least squares fit of Equation 6 to empirical data is equivalent to a two-independent-variable multiple linear regression in which the constant term is forced to zero, and the weighting coefficients k_3 and k_4 play the role of regression weights. In the present study these weights were determined for each subject individually by regression of inspiratory/expiratory signal differences to expired breath-volumes.

During the calibration session subjects sat relaxed in a supine position, two belts attached around chest and abdomen, and they exhaled into a mechanical spirometer.[10] After each breath the exhaled volume was read off and the display was reset. Following a period of normal breathing they were instructed to take some deeper or quieter breaths and care was taken to obtain about 30 breaths covering a range from .5 to over 2 liters.

The approximately 30 exhaled volumes from this session were taken as a sample of the criterion variable. To obtain the corresponding samples of the independent variables, from a strip chart of the circumference signals, the distances from the inhalation maxima to the next exhalation minima were determined (these readings being in millimeters). With these data a least squares solution of Equation 6 was calculated using a Fortran program specially written for this purpose.

Table 2 gives results of the calibration procedure for 15 subjects. Comparing the combined correlation R with single correlations shows an improvement in every case (as is implicit in multiple regression). Predictions of expired volume range from 50

[10]The spirometer was a mechanical "Draeger Volumeter." A continuously measuring device would have been superior but was not at our disposal.

Table 2

Regression weights and correlations between predicted and measured expired volumes

Subject No.	Regression Weights: Chest	Regression Weights: Abdomen	Correlation of Predicted and Measured Volumes (R)	Relative Mean Squared Error[a] %	No. of Exhalations (n)	Single Correlations with Volume: Chest	Single Correlations with Volume: Abdomen
1	−4.9	319.5	.9752	5.6	50	.6430	.9746
2	16.0	10.8	.9623	8.2	13	.9435	.5657[b]
3	−0.2	21.9	.8893	23.5	17	.8328	.8893
4	12.4	−4.6	.7068	52.1	26	.7176	−.3237[c]
5	26.8	4.4	.8883	22.6	21	.8821	.9317
6	14.4	15.2	.9002	39.6	22	.7204	.8011
7	−4.8	20.9	.9101	18.0	26	.4299	.9122
8	31.6	24.7	.9536	26.9	29	.8326	.9357
9	18.9	42.5	.8798	41.3	39	.7750	.7367
10	45.9	−19.1	.8226	33.7	31	.8249	.1170
11	74.8	11.1	.8515	28.6	36	.8386	.6911
12	28.2	42.5	.8271	33.3	27	.8008	.7515
13	47.7	68.1	.8850	23.4	15	.8439	.8632[d]
14	16.7	15.6	.9649	7.2	23	.9432	.7212
15	36.1	10.1	.9533	9.7	22	.9505	.7373

[a]Due to the modified regression model (see text), the relative mean squared error is not equal to $(1-R^2)$.

[b]This case demonstrates that very few exhalations might suffice for calibration purposes.

[c]This subject was very small and had a low tidal volume. Below 50 ml the mechanical spirometer did not function well, so the standard deviation of volume was only 139 ml, resulting in comparably high error variance.

[d]This subject had a very high tidal volume, and the standard deviation of volume was 715 ml.

to 95% (100% minus relative mean squared error in column 5). Figure 4 shows a sample distribution of amplitudes of predicted and measured breath-volumes for one subject. Note the linearity of prediction.

The linear regression analysis used is modified from the standard model because least square optimization has to be done for a function with missing constant term (a result of AC-coupling). Therefore some of the basic relationships no longer hold: Relative error variance, for instance, is no longer equal to $1 - R^2$ (see Appendix B).

The modification of the regression model is necessitated by the fact that the prediction of a continuous variable is attempted with model parameters being optimized for the sampled variables; and the set of samples is not representative of the (infinite) population of continuous values because samples have to be taken at the extreme values of the volume-time functions[11] (see Footnote 10). Therefore, with the calibration performed with a continuously measuring spirometer as criterion, still better results would be expected. In the latter case a standard two-independent-variable regression would be equivalent to fitting Equation 6 since the constant term would go to zero. (The constant term is composed of a weighted sum of the means of all three variables, and, considering sufficiently long time intervals, these means are zero due to AC-coupling.)

[11]This sampling should not be confused with equidistant sampling used in digital signal processing. Unlike the latter, it is *not* independent of the signal and therefore does not show aliasing errors.

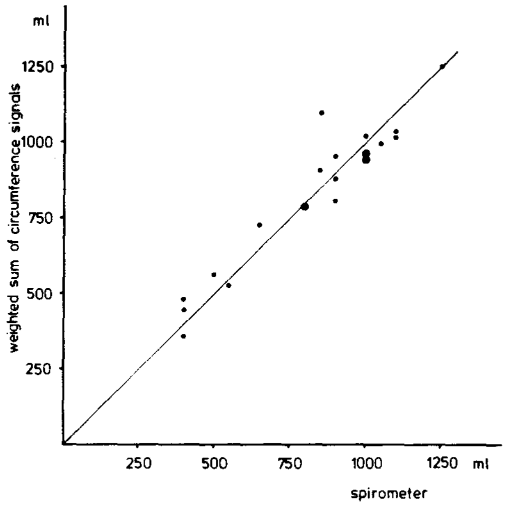

Figure 4. Distribution of expired volumes predicted from summed circumference signals vs measured volumes for one subject.

Because of the high correlation between the two independent variables, quite different combinations of regression coefficients can lead to similar predicted values. Hence regression weights tend to be sensitive to small variations of experimental setup, but this sensitivity does not affect the accuracy of volume prediction. A small weight is an indication that the corresponding independent variable has a smaller correlation with the criterion than the other variable, and in this way does less for prediction. In cases where one weight is negative, the corresponding coefficient in the summing circuit is set to zero, so only the circumference signal with the better volume-correlation remains. Still in these cases the use of two transducers is superior to the use of a single one because it is not known in advance which one will yield the better correlation.

The determined weights were used for an analog real-time measure of breath-volume. On-line calculation of Equation 6 was done with a simple summing circuit on an EAI 380 analog computer with individual coefficients (potentiometer settings) k_3 and k_4 set to the determined regression weights. Finally, it should be mentioned that regression weights bear no relationship to the relative proportion that abdomen and chest motions contribute to breath-volume. A different procedure might try to assess this proportion and take it as a basis for the summing weights (for example, Shapiro & Cohen, 1965).

In summary, the following procedure proved to be successful for application in other studies: Two belts are applied in the usual way, one to measure thoracic and the other abdominal circumference, and in a short calibration session individual regression weights are determined. These weights are used as coefficients for an on-line summing amplifier. In cases where one weight is small or negative the corresponding coefficient is set to zero, so only the circumference signal with the better volume-correlation remains. Using this procedure yields an accuracy that is not obtainable with a single belt or other indirect volume measurement methods.

Application to Paced Respiration

The described two-belt respiration measuring technique has proven well suited to the requirements of paced respiration. For pacing, the subjects are shown their own respiration signal on a display and they are instructed to keep it inside a "window" around a reference signal. Figure 5 shows the efforts of an untrained subject to perform square and sinusoidal respiration patterns. Figure 6 shows the average of 10 trials of step-inspiration and expiration each followed by 20 sec of breath-holding

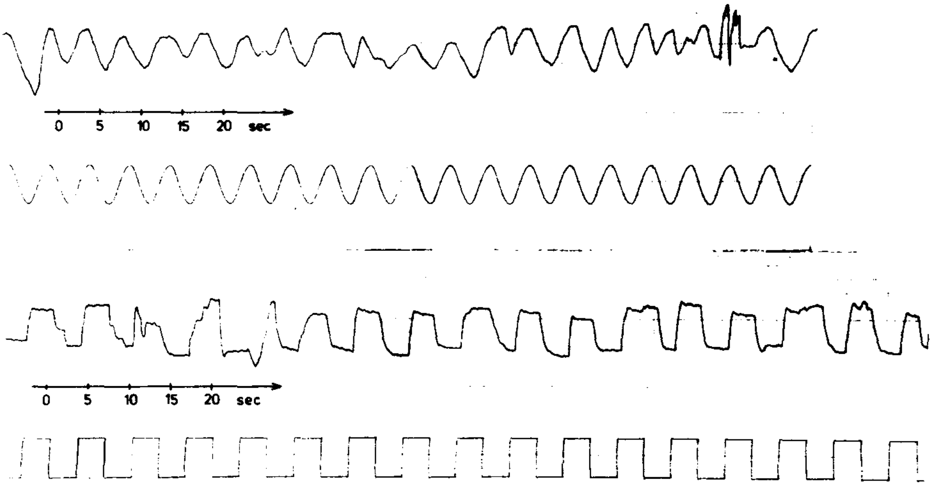

Figure 5. Recording of paced respiration. **a)** Sinus-pattern. **b)** Square pattern.

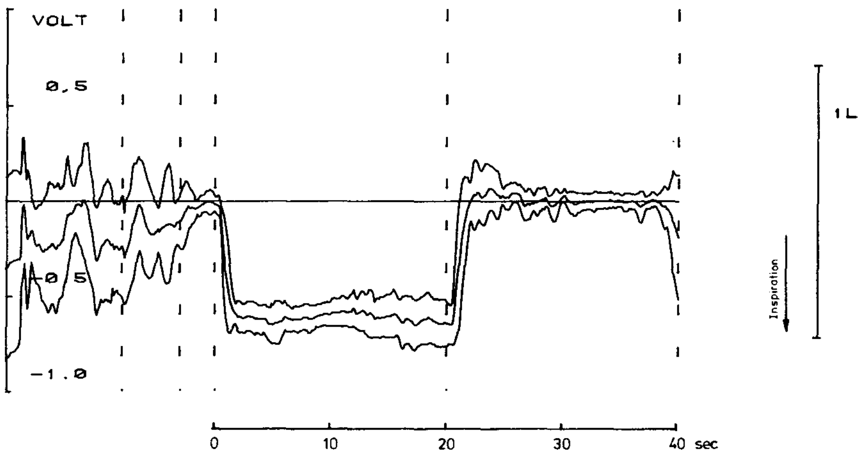

Figure 6. Average over 10 trials of step-inspiration—20-sec-breath-holding—step-expiration—20-sec-breath-holding (middle line), together with the standard deviation (distance of upper and lower line to middle line). The signal to the left of zero time represents the last part of free breathing periods. Note: 5%-confidence-interval amounts to 62% of shown standard deviation.

(middle line), together with standard deviation (distance of upper and lower line to middle line). Note the constancy of the averaged signal and standard deviation during the long breath-holding interval obtainable with our measuring device. Note further the sharp edges of inhalation and exhalation demonstrating the subject's ability to quickly and accurately respond to the pacing signal. This task is particularly demanding on the measurement method's precision.

Use of a highpass filter (as shown in Figure 3) even with time-constants exceeding 50 sec is not recommended for a pacing application. With AC-coupling the signal is stabilized in a way that the mean square signal above and below zero becomes equal. Respiration patterns where unequal amounts of time are spent during expiration and inspiration make the minima of the physiological signal, i.e. relaxed respiration state, unequal to the minima of the trace on the display. But for the subjects these minima are a more important baseline than the mean of the signal, and our subjects found this discrepancy between displayed baseline and the relaxed expiration state very confusing. Thus, true DC-recording is essential for the subject's performance of paced respiration tasks.

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Appendix A

Ignoring second-order terms

From Equation 4, the error introduced by ignoring second-order terms is $e = B/(A+B)$, where B is the sum of second-order terms $B = \Delta C_1^2+\Delta C_2^2+\Delta C_1\Delta C_2$ and A is the sum of the remaining linear terms. Since squares are always positive this error is smaller than $e_1 = B/A$ which is

$$e_1 = \frac{\Delta C_1^2+\Delta C_2^2+\Delta C_1\Delta C_2}{(2C_1+C_2)\Delta C_1+(2C_2+C_1)\Delta C_2}.$$

Let C be the smaller of the circumferences, then this expression is smaller than

$$e_2 = \frac{(\Delta C_1+\Delta C_2)^2-\Delta C_1\Delta C_2}{3C(\Delta C_1+\Delta C_2)}.$$

Assume without loss of generality that C_1 is smaller than C_2 and all values are positive, then e_2 is smaller than

$$e_3 = \frac{\Delta C_1}{3C} + \frac{\Delta C_2}{3C} - \frac{\Delta C_1\Delta C_2}{3C(2\Delta C_2)}$$

$$= \frac{\Delta C_1}{6C} + \frac{\Delta C_2}{3C}$$

$$\leq ½\frac{\Delta C_2}{C}.$$

If one is not interested in assessing the error introduced by linearizing, a much simpler argument leads to Equation 6. Starting from Equation 3 the volume increment for small changes in the independent variables is given by the total differential dV:

$$dV = \frac{\partial V}{\partial C_1}dC_1 + \frac{\partial V}{\partial C_2}dC_2.$$

The partial derivatives are

$$\frac{\partial V}{\partial C_i} = \frac{h}{12\pi}(2C_i+C_j)$$

which contain only linear terms.

Appendix B

Modification of regression model

The standard two-independent-variable regression model is

$$z = \mathrm{a}\cdot x + \mathrm{b}\cdot y + \mathrm{c} \tag{1}$$

with constants a, b, and c. A least squares fit of this equation to empirical data leads to the normal equations E_i which determine the regression coefficients a, b, c. With these coefficients, for each set of data points (x_i,y_i) a predicted value $\hat{z}_i$ can be calculated. Usual criteria of goodness of fit are:

the correlation R between $\hat{z}$ and z,

the variance $s_{\hat{z}}^2$ of $\hat{z}$, called the "accounted for variance,"

the error variance s_e^2, i.e. the variance of the error variable $(z-\hat{z})$,

the mean squared error

$$MS_e = \frac{\sum_i^n (z_i-\hat{z}_i)^2}{n-3}$$

where n is the number of observations.

Error variance and mean squared error are connected by the relationship

$$s_e^2 = MS_e + \bar{e}^2,$$

where $\bar{e}$ is the mean prediction error (i.e. the mean of the error variable $(z-\hat{z})$). Since the mean error from predictions by Equation 1 is zero, error variance and mean squared error are equal.

The general decomposition of variances is

$$s_z^2\cdot(n-1) = (s_e^2 + s_{\hat{z}}^2 + 2s_{e\hat{z}}^2)(n-3) \tag{2}$$

where $s_{e\hat{z}}^2$ is the covariance between error and prediction. For a least squares fit of Equation 1 it can be shown that this covariance is zero, so (for sufficiently large n) error variance and prediction variance "add up" to the variance s_z^2 of the criterion variable z. This is the justification to call $s_{\hat{z}}^2$ the "variance accounted for." The accounted proportion of variance $s_{\hat{z}}^2/s_z^2$ can be shown to be equal to R^2, and sometimes this equality is used as a definition of R. Another alternative way of defining the multiple correlation coefficient R is in terms of the raw data, which leads to an interpretation of R as a measure of linear connectivity between the three variables x, y and z.

Now, fitting an equation with a zero constant term c

$$z = \mathrm{a}\cdot x + \mathrm{b}\cdot y \tag{3}$$

leads to a new set of normal equations for the coefficients which are different from the equations obtained for Model 1:

$$b = \frac{XX \cdot YZ - XY \cdot XZ}{XX \cdot YY - XY \cdot XY} \quad (4)$$

$$a = \frac{XZ - b \cdot XY}{XX} \quad (5)$$

where

$$XX = \Sigma x_i^2 \qquad YY = \Sigma y_i^2$$

$$XY = \Sigma x_i y_i \qquad XZ = \Sigma x_i z_i \qquad YZ = \Sigma y_i z_i$$

These equations can be derived from the usual normal equations E_i by setting in them the means of all variables to zero.

The fit of Equation 3 is suboptimal as a two-independent-variable model, and as a consequence

(6) the mean prediction error is not zero, i.e. the means of z and $\hat{z}$ are different,

(7) the covariance $s_{e\hat{z}}^2$ is not zero.

As a consequence of Statement 6,

(8) error variance is not equal to mean squared error and as a consequence of Statement 7,

(9) the three definitions of the multiple correlation coefficient R given above lead to different results; especially when R is defined as the correlation between z and $\hat{z}$, then $1-R^2$ is not equal to error variance or mean squared error.

Although the fit of Equation 3 is suboptimal for the sampled data, it is mandatory for predicting continuous data. For continuous data, the means of all variables are zero due to either AC-coupling or offset correction. Adding a constant term c to the prediction signal means adding a constant DC-signal which has to be removed again. In effect, then, the predicted amplitude is too small by an amount of c which augments the mean squared error by c^2.

Criterion of fit

Although for optimal fit the multiple correlation reaches its maximum and error variance reaches its minimum, neither are useful as criteria for determining the regression weights. This is because both are invariant with shifting the prediction by a constant e (i.e. changing the constant term in the regression equation), whereas the mean squared error is increased by e^2. Furthermore, the correlation coefficient is invariant with scaling the whole equation up or down. So if the regression weights were to be determined directly by an on-line circuit on the analog computer calculating the criterion of fit in real-time, neither the correlation coefficient, nor the error variance may be used. Instead, the error sum of squares would be useful for example.